```
Clear["Global`*"]
```

1. Normal distribution. Apply the maximum likelihood method to the normal distribution with $\mu = 0$.

What did I do here? I ginned up a list of outcomes using the normal distribution with the mean required by the problem description and a small standard deviation. Then I asked Mathematica to create a distribution based on the characteristics of the list members and the **`ParameterEstimator`** using the “MaximumLikelihood” method. I can view the distribution created and judge its fidelity to the original.

```
data = RandomVariate[NormalDistribution[0, 0.001], 10000];
EstimatedDistribution[data, NormalDistribution[n, p],
 ParameterEstimator → {"MaximumLikelihood"}]
```

```
NormalDistribution[5.67086 × 10^-6, 0.00100286]
```

I notice that each time I re-run the above command the output changes slightly.

The **`ParameterEstimator`** procedure has a half dozen search methods, including

```
EstimatedDistribution[data, NormalDistribution[n, p],
 ParameterEstimator → {"MethodOfMoments"}]
```

```
NormalDistribution[5.67086 × 10^-6, 0.00100286]
```

```
EstimatedDistribution[data, NormalDistribution[n, p],
 ParameterEstimator → {"MethodOfCumulants"}]
```

```
NormalDistribution[5.67086 × 10^-6, 0.00100286]
```

I could plot the pdfs of the three manufactured distributions but there would be nothing to distinguish one from another.

3. Poisson distribution. Derive the maximum likelihood estimator for μ. Apply it to the sample (10, 25, 26, 17, 10, 4), giving numbers of minutes with 0 - 10, 11 - 20, 21 - 30, 31 - 40, 41 - 50, more than 50 fliers per minute, respectively, checking in at some airport check-in.

```
Clear["Global`*"]
```

I was able to hit the right buttons on this one, I think.

```
data = {10, 25, 26, 17, 10, 4}
```

```
{10, 25, 26, 17, 10, 4}
```

```
FindDistributionParameters[data, PoissonDistribution[μ],
 ParameterEstimator → {"MaximumLikelihood", Method → "NMaximize"}]
```

```
{μ → 15.3333}
```

The answer above matches the answer in the text.

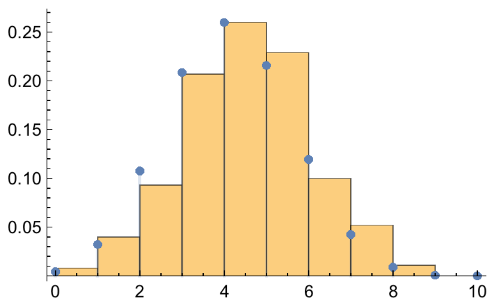
5. Binomial distribution. Derive a maximum likelihood estimate for p.

```
Clear["Global`*"]
```

```
data = RandomVariate[BinomialDistribution[10, 0.4], 1000];
dink = EstimatedDistribution[data, BinomialDistribution[n, p],
  ParameterEstimator → {"MaximumLikelihood"}]
```

```
BinomialDistribution[9, 0.453778]
```

```
Show[Histogram[data, {0, 10, 1}, "PDF"], DiscretePlot[PDF[dink, x],
  {x, 0, 11}, PlotStyle → PointSize[Medium]], ImageSize → 250]
```



If the maximum likelihood is at least nominal, as here, I would have to consider it as reasonably likely to represent the goal of the exercise.

```
data1 = RandomVariate[BinomialDistribution[10, 0.4], 10 000];
dink1 = EstimatedDistribution[data1, BinomialDistribution[n, p],
  ParameterEstimator → {"MaximumLikelihood"}]
```

```
BinomialDistribution[10, 0.40283]
```

```
data2 = RandomVariate[BinomialDistribution[10, 0.4], 30 000];
dink2 = EstimatedDistribution[data2, BinomialDistribution[n, p],
  ParameterEstimator → {"MaximumLikelihood"}]
```

```
BinomialDistribution[10, 0.400487]
```

Above I see the natural tendency for the maximum likelihood to go down as the sample size increases.

6. Extend problem 5 as follows. Suppose that m times n trials were made and in the first n trials A happened k_1 times, in the second n trials A happened k_2 times, . . . , in the mth n trials A happened k_m times. Find a maximum likelihood estimate of p based on this information.

7. Suppose that in problem 6 we made 3 times 4 trials and A happened 2, 3, 2 times,

respectively. Estimate p.

```
Clear["Global`*"]
```

```
data7 = RandomVariate[BinomialDistribution[12, 7/12], 100 000];
```

```
dink71 = EstimatedDistribution[data7,
  BinomialDistribution[n, p], ParameterEstimator → {"MethodOfMoments"}]
```

```
BinomialDistribution[12.0097, 0.582034]
```

```
dink72 = EstimatedDistribution[data7, BinomialDistribution[n, p],
  ParameterEstimator → {"MaximumLikelihood"}]
```

```
BinomialDistribution[12, 0.582506]
```

```
dink73 = EstimatedDistribution[data7, BinomialDistribution[n, p],
  ParameterEstimator → {"MethodOfCentralMoments"}]
```

```
BinomialDistribution[12.0097, 0.582034]
```

```
dink74 = EstimatedDistribution[data7, BinomialDistribution[n, p],
  ParameterEstimator → {"MethodOfCumulants"}]
```

```
BinomialDistribution[12.0097, 0.582034]
```

```
dink75 = EstimatedDistribution[data7, BinomialDistribution[n, p],
  ParameterEstimator → {"MethodOfFactorialMoments"}]
```

```
BinomialDistribution[12.0097, 0.582034]
```

```
N[7 / 12]
```

```
0.583333
```

A couple of comments. I notice that **RandomVariate** will accept a rational fraction as standard deviation. As for the given information, I don't think the number of trials matters, I believe that only the raw ratio is important here. After many run-throughs of testing, I see the following: Concerning the **RandomVariate**, the sample size, the last number, can't smooth out the sample if the first number, trials, has small precision. This makes sense, though with problem 7 it is a fly in the ointment. As for the dinks, the **MaximumLikelihood** always differs from the other methods, though not always on the high side of the theoretical ratio decimal value. The other four methods, in my experience, always come up with exactly the same answer on this problem.

8. Geometric distribution. Let X = Number of independent trials until an event A occurs. Show that X has a geometric distribution, defined by the probability function f[x] = pq^{x-1}, x = 1, 2, . . . , where p is the probability of A in a single trial and q = 1 - p. Find the maximum likelihood estimate of p corresponding to a sample x_1 , x_2 , . . . , x_n of observed values of X.

9. In problem 8, show that f(1) + f(2) + . . . + = 1 (as it should be). Calculate indepen-

dently of problem 8 the maximum likelihood of p in problem 8 corresponding to a single observed value of X.

```
Clear["Global`*"]
```

```
inr = PDF[GeometricDistribution[p], k]
```

$$\begin{cases} (1-p)^k p & k \geq 0 \\ 0 & \text{True} \end{cases}$$

```
tri = Sum[inr, {k, 0, 10}]
```

$$p + (1-p)\,p + (1-p)^2 p + (1-p)^3 p + (1-p)^4 p + (1-p)^5 p + (1-p)^6 p + (1-p)^7 p + (1-p)^8 p + (1-p)^9 p + (1-p)^{10} p$$

```
tri /. p → 0.8
```

```
1.
```

The pdf shown above is equivalent to the one described in problem 8.

```
data9 = RandomVariate[GeometricDistribution[0.1], 1]
```

```
{3}
```

```
dib = EstimatedDistribution[data9, GeometricDistribution[n],
  ParameterEstimator → {"MaximumLikelihood"}]
```

```
GeometricDistribution[0.25]
```

Observations on this problem. **RandomVariate** and the **GeometricDistribution** will not produce a non-zero single sample unless the input probability is 0.4 or less. If X is the sample, then the output probability is approximately $\frac{1}{X}$, which is the text answer for this problem.

11. Find the maximum likelihood estimate of θ in the density $f[x] = \theta\, e^{-\theta x}$ if $x \geq 0$ and $f[x] = 0$ if $x < 0$.

```
Clear["Global`*"]
```

The following is not the way to define the desired distribution, because Mathematica can't understand this formatting.

```
dist = ProbabilityDistribution[Piecewise[{{θ e^(-θ x), x ≥ 0}, {0, x < 0}}],
   {x, 0, ∞}, Assumptions → {θ > 0}];
```

```
PDF[dist[], x]
```

$$\text{PDF}\left[\text{ProbabilityDistribution}\left[\begin{cases} e^{-x\,\theta}\,\theta & x \geq 0 \\ 0 & \text{True} \end{cases}, \{x, 0, \infty\}, \text{Assumptions} \to \{\theta > 0\}\right][\,], x\right]$$

As an improvement on the above, the following was tried thanks to an example by Bob Hanlon on *https://mathematica.stackexchange.com/questions/72996/custom-distribution*

```
dist2 = ProbabilityDistribution[θ ⅇ^(-θ x), {x, 0, ∞}, Assumptions → {θ > 0}]
```

```
ProbabilityDistribution[ⅇ^(-x θ) θ, {x, 0, ∞}, Assumptions → {θ > 0}]
```

For Mathematica to return the pdf in processed form is a good sign.

```
PDF[dist2, x]
```

$$\begin{cases} e^{-x\theta}\theta & x > 0 \\ 0 & \text{True} \end{cases}$$

```
dpa = DistributionParameterAssumptions[dist2]
```

```
{θ > 0}
```

The pdf turns out to be normalized.

```
Assuming[dpa, Integrate[PDF[dist2, x], {x, 0, Infinity}]]
```

```
1
```

```
Assuming[dpa, Mean[dist2] // FullSimplify]
```

$$\frac{1}{\theta}$$

```
Assuming[dpa, StandardDeviation[dist2] // Simplify]
```

$$\frac{1}{\theta}$$

With mean equal to s.d. , I want to try the following.

```
PDF[dist2, x] == PDF[ExponentialDistribution, x] // Simplify[#, dpa] &
```

$$\text{PDF[ExponentialDistribution, x]} == \left(\begin{cases} e^{-x\theta}\theta & x > 0 \\ 0 & \text{True} \end{cases}\right)$$

The above cell does not prove that dist2 is exponential, but in appearance it is exponential. I can choose θ to be 0.6 just for grins and try out a **`MaximumLikelihood`** move.

```
data11 = RandomVariate[ExponentialDistribution[0.6], 100 000];
```

```
EstimatedDistribution[data11, ExponentialDistribution[p],
 ParameterEstimator → {"MaximumLikelihood"}]
```

```
ExponentialDistribution[0.601181]
```

This is not what the text answer looks like, that being $\hat{\theta} = \frac{1}{\bar{x}}$.

13. Compute $\hat{\theta}$ in problem 11 from the sample 1.9, 0.4, 0.7, 0.6, 1.4. Graph the sample distribution function $\hat{F}$[x] and the distribution function F[x] of the random variable, with $\theta = \hat{\theta}$, on the same axes. Do they agree reasonably well? (We consider goodness of fit systematically in section 25.7.)

```
myemp = EmpiricalDistribution[{1.9, 0.4, 0.7, 0.6, 1.4}]
```

```
DataDistribution[ Type Empirical
                  Datapoints 5
                  Inputdimension 1
                  Domain {0.4, 1.9} ]
```

```
data13 = {1.9, 0.4, 0.7, 0.6, 1.4}
```

```
{1.9, 0.4, 0.7, 0.6, 1.4}
```

```
p1 = DiscretePlot[CDF [myemp, x],
    {x, -4, 4, .01}, PlotStyle → Red, ImageSize → 250];
```

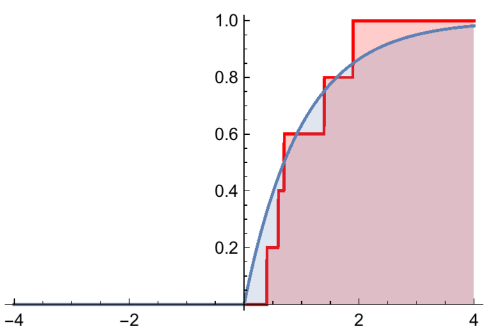
```
sec = EstimatedDistribution[data13, ExponentialDistribution[p],
  ParameterEstimator → {"MaximumLikelihood"}]
```

```
ExponentialDistribution[1.]
```

The above answer agrees (accidentally?) with the text answer.

```
p2 = DiscretePlot[CDF [sec, x], {x, -4, 4, .01}];
```

```
Show[p1, p2]
```



15. CAS EXPERIMENT. Maximum likelihood estmates. (MLEs). Find experimentally how much MLEs can differ depending on the sample size. Hint. Generate many samples of the same size n, e.g. of the standardized normal distribution, and record $\bar{x}$ and s^2. Then increase n.

This is indeed an experiment. If the cells were to be run again, all answers would be different. It demonstrates that sample size is limited in influence so long as the mean and standard deviation are expressed coarsely.

```
Clear["Global`*"]
```

```
data15a = RandomVariate[NormalDistribution[0.2, 0.01], 4];
data15b = RandomVariate[NormalDistribution[0.2, 0.01], 12];
data15c = RandomVariate[NormalDistribution[0.2, 0.01], 100];
data15d = RandomVariate[NormalDistribution[0.2, 0.01], 1000];
data15e = RandomVariate[NormalDistribution[0.2, 0.01], 10 000];
data15f = RandomVariate[NormalDistribution[0.2, 0.01], 100 000];

d15a = EstimatedDistribution[data15a, NormalDistribution[n, p],
  ParameterEstimator → {"MaximumLikelihood"}]
NormalDistribution[0.197877, 0.0105572]

d15b = EstimatedDistribution[data15b, NormalDistribution[n, p],
  ParameterEstimator → {"MaximumLikelihood"}]
NormalDistribution[0.202904, 0.00793746]

d15c = EstimatedDistribution[data15c, NormalDistribution[n, p],
  ParameterEstimator → {"MaximumLikelihood"}]
NormalDistribution[0.199636, 0.00986493]

d15d = EstimatedDistribution[data15d, NormalDistribution[n, p],
  ParameterEstimator → {"MaximumLikelihood"}]
NormalDistribution[0.199854, 0.00995169]

d15e = EstimatedDistribution[data15e, NormalDistribution[n, p],
  ParameterEstimator → {"MaximumLikelihood"}]
NormalDistribution[0.199871, 0.0100522]

d15f = EstimatedDistribution[data15f, NormalDistribution[n, p],
  ParameterEstimator → {"MaximumLikelihood"}]
NormalDistribution[0.199941, 0.00994559]

n1 = Abs[0.2 - d15a[[1]]];
n2 = Abs[0.01 - d15a[[2]]];
n3 = Abs[0.2 - d15b[[1]]];
n4 = Abs[0.01 - d15b[[2]]];
n5 = Abs[0.2 - d15c[[1]]];
n6 = Abs[0.01 - d15c[[2]]];
n7 = Abs[0.2 - d15d[[1]]];
n8 = Abs[0.01 - d15d[[2]]];
n9 = Abs[0.2 - d15e[[1]]];
n10 = Abs[0.01 - d15e[[2]]];
n11 = Abs[0.2 - d15f[[1]]];
n12 = Abs[0.01 - d15f[[2]]];

g2 = Grid[
   {{" trial ", "sample", " error mean ", " error s.d. "}}, Frame → All];
```

```
g1 = Grid[{{"data15a", 4, n1, n2}, {"data15b", 12, n3, n4},
    {"data15c", 100, n5, n6}, {"data15d", 1000, n7, n8}, {"data15e",
     10 000, n9, n10}, {"data15f", 100 000, n11, n12}}, Frame → All];
```

```
Column[{g2, g1}]
```

trial	sample	error mean	error s.d.
data15a	4	0.00212348	0.000557249
data15b	12	0.00290421	0.00206254
data15c	100	0.00036434	0.00013507
data15d	1000	0.000146174	0.0000483148
data15e	10 000	0.000128969	0.0000522355
data15f	100 000	0.0000593452	0.0000544065

In the above grid it can be seen that the results of the **MaximumLikelihood** search get closer to the nominal values as the samples get larger.